Event-Based Control of Nonlinear Systems with Partial State and Output Feedback*

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Abstract

This paper studies the event-triggered control problem for nonlinear systems with partial state and output feedback. We first consider the control systems that are transformable into an interconnection of two input-to-state stable (ISS) subsystems with the sampling error as the external input. It is shown that infinitely fast sampling can be avoided and asymptotic stabilization can be achieved by appropriately choosing the decreasing rate of the threshold signal of the event-trigger. Then, we focus on the event-triggered output-feedback control problem for nonlinear uncertain systems in the output-feedback form. The key idea is to introduce a novel nonlinear observer-based control design and to transform the control system into the form of interconnected ISS systems. ISS small-gain methods are used as a fundamental tool in the discussions.

Keywords

Event-triggered control, nonlinear systems, partial state feedback, output feedback, input-to-state stability (ISS), small-gain theorem.

1 Introduction

Tremendous effort has been made for improved performance of sampled-data control systems. As an alternative to the traditional periodic data-sampling, the aperiodic event-triggered data-sampling depends on the real-time system state, and in this way, takes into account the system behavior between the sampling time instants. Such new data-sampling strategy has been proved to be quite useful in reducing the waste of computation and communication resources in

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feedback control systems. Early results in this direction include [14, 31, 49, 5, 41].

Due to the increasing popularity of networked control systems, recent years have seen a renewed interest in event-triggered control of linear and nonlinear systems. Significant contributions have been made to the literature; see, e.g., [2, 4, 51, 46, 18, 19, 10, 38, 7, 39, 47] and the references therein. The designs have been extended to distributed control in [50, 8, 43, 13], decentralized control in [7, 6], systems with quantized measurements in [9] and periodic event-triggered control in [16], to name a few. See also [17] for a literature review and tutorial of event-triggered control,

For practical implementation of event-triggered control, infinitely fast sampling should be avoided, that is, the intervals between all the sampling time instants should be lower bounded by some positive constant; see e.g., the discussions in [29]. One special case of infinitely fast sampling is the Zeno behavior, i.e., there is an infinite number of sampling time instants which converge to a finite value; see, e.g., [12] for the discussions on the Zeno behavior in hybrid systems. In most of the existing results, the events of data-sampling are triggered by comparing the real-time system state and a threshold signal, and the event-triggered control problems are transformed into problems of choosing appropriate threshold signals to avoid infinitely fast sampling. Due to the hybrid nature, the forward completeness of the event-triggered control systems is a complex issue.

This paper focuses on the event-triggered control problem for nonlinear systems with partial state and output feedback. Several recent results can be found in [3, 42, 7, 15, 30]. Specifically, in [3, 42, 15, 30], observers are employed to reconstruct the plant state by using the sampled data of the outputs. The paper of [7] studies general dynamic output-feedback controllers and uses the theory of hybrid systems to analyze the stability property of the resulted closed-loop systems. In [28], a prediction mechanism is employed in the controller to estimate the system output between the sampling time instants, which can be considered as the output-feedback version of [38]. Note that the event-triggered output-feedback control results mentioned above mainly focus on linear systems. Also, it seems that most of the existing results on partial state and output feedback event-based control for continuous-time, deterministic systems can only guarantee practical convergence.

The notion of input-to-state stability (ISS), invented by Sontag, is a powerful tool to describe the stability property of nonlinear systems with external inputs; see [44]. For event-triggered control, ISS has been used to describe the influence of data-sampling to control system performance; see, e.g., [46, 1, 40]. In this framework, it is often assumed that the plant has a known input-to-state stabilizing controller with the sampling error considered as the external input. Then, asymptotic stabilization can be achieved if one can find an event-trigger such that the influence of the sampling error is attenuated and the closed-loop system augmented with the event-trigger is asymptotically stable at the origin. In the very recent paper [6], the ISS small-gain theorem in [24] is applied to guarantee the stability of the overall system composed of interacting ISS subsystems, and a parsimonious event triggering mechanism is developed to avoid the Zeno behavior. In our recent paper [34], we proposed an ISS gain condition for event-triggered control of nonlinear uncertain systems with full-state feedback without using Lyapunov arguments.

This paper takes a step forward toward solving the event-triggered control problem for nonlinear systems with partial state and output feedback. The contribution of the paper is twofold:

- A novel ISS small-gain approach to event-trigger design is proposed for asymptotic convergence of interconnected nonlinear systems with partial state and output feedback;
- A constructive event-triggered output-feedback control design is developed for a class of nonlinear uncertain systems.

The problems are studied in this paper by using the Lyapunov-based ISS arguments. To the best of our knowledge, both of the contributions are new. At the same time, this paper takes into full account the forward completeness issue caused by the hybrid dynamics of event-triggered control systems.

We first consider the control systems that are transformable into an interconnection of two input-to-state stable (ISS) subsystems with the sampling error as the external input. It is also assumed that only the state of one subsystem is available for event-trigger design. Motivated by [13, 43], we propose an event-trigger design method with the threshold signal generated by an asymptotically stable system. By considering the closed-loop event-triggered system as a dynamic network, we use the ISS small-gain methods developed in [24, 22, 25, 32] to fine-tune the parameters of the event-trigger to avoid infinitely fast sampling, and at the same time, guarantee asymptotic convergence of the system state. It should be noted that, in [13, 43], exponentially converging threshold signals are used. But, with an elementary example in this paper, we show that non-exponentially converging threshold signals may be preferred for nonlinear systems. As a special case, if the subsystems are linear with quadratic ISS-Lyapunuov functions, one can find exponentially decreasing threshold signals. The new design has also been extended for the first time to a more general case in which the dynamics of the threshold signal subsystem also depends on the available state information of the controlled system. Note that this paper uses Lyapunov-based formulations and focuses on the problems caused by partial state and output feedback, which differs from our recent paper [34] focusing on full state feedback event-triggered and self-triggered control.

The second contribution of the paper lies in a new event-triggered output-feedback control design for nonlinear uncertain systems in the output-feedback form; see [27]. The proposed controller is composed of a novel ISS-induced nonlinear observer and a nonlinear control law. By appropriately choosing the parameters, the controlled system can be ultimately transformed into an interconnection of two ISS subsystems with the sampling error of the output as the external input. Then, an event-trigger design is proposed for output-feedback stabilization.

The rest of the paper is organized as follows. Section 2 presents the notations and reviews ISS and a cyclic-small-gain result that will be used for the designs in this paper. In Section 3, we propose a new event-trigger design for a class of control systems transformable into an interconnection of two ISS subsystems. In Section 4, we focus on the event-triggered output-feedback control problem of nonlinear uncertain systems in the output-feedback form. A simulation example is given in Section 5 to show the need of a non-exponentially decreasing threshold signal for a nonlinear system. Section 6 contains some concluding remarks.

2 Preliminaries

To make the paper self-contained, recall that a function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is positive definite if $\gamma(s) > 0$ for all $s > 0$ and $\gamma(0) = 0$. $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{K}$ function (denoted by $\gamma \in \mathcal{K}$) if it is continuous, strictly increasing and $\gamma(0) = 0$; it is a $\mathcal{K}_\infty$ function (denoted by $\gamma \in \mathcal{K}_\infty$) if it is a $\mathcal{K}$ function and also satisfies $\gamma(s) \to \infty$ as $s \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathcal{KL}$ (denoted by $\beta \in \mathcal{KL}$) if $\beta(\cdot, t)$ is a $\mathcal{K}$ function for each fixed t and $\beta(s, t)$ is decreasing to zero as $t \to \infty$ for each $s \in \mathbb{R}_+$. Id represents the identity function. For $\gamma_1, \gamma_2 \in \mathcal{K}$, inequality $\gamma_1 \circ \gamma_2 < \mathrm{Id}$ means $\gamma_1(\gamma_2(s)) < s$ for all $s > 0$.

In this paper, we use $\nabla f(x)$ to represent the gradient of function f at x if f is differentiable at x. For functions f_1, f_2, we define $\partial f_1(f_2(x)) = \nabla f_1(x')|_{x'=f_2(x)}$ for all the x such that $\nabla f_1(x')$ exists when $x' = f_2(x)$.

The ISS cyclic-small-gain theorem proposed in [48, 25, 32] is a tool to analyze the ISS property of large-scale nonlinear systems in the following form:

$$\dot{x}_i = f_i(x, u_i), \quad i = 1, \ldots, N \tag{1}$$

where $x_i \in \mathbb{R}^{n_i}$, $x = \left[x_1^T, \ldots, x_N^T\right]^T$, $u_i \in \mathbb{R}^{m_i}$ and $f_i : \mathbb{R}^{n+m_i} \to \mathbb{R}^{n_i}$ with $n = \Sigma_{j=1}^N n_j$ is a locally Lipschitz map.

It is assumed that each x_i-subsystem ($i = 1, \ldots, N$) in (1) admits an ISS-Lyapunov function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}$ which is locally Lipschitz on $\mathbb{R}^{n_i} \backslash \{0\}$ and satisfies

1. there exist $\underline{\alpha}_i, \overline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|x_i|) \le V_i(x_i) \le \overline{\alpha}_i(|x_i|), \quad \forall x_i \in \mathbb{R}^{n_i}; \tag{2}$$

2. there exist $\gamma_{ij} \in \mathcal{K} \cup \{0\}$ ($j \neq i$) and $\gamma_i^u \in \mathcal{K} \cup \{0\}$ such that

$$\begin{aligned} &V_i(x_i) \ge \max_{j=1,\ldots,N; j\neq i} \left\{\gamma_{ij}(V_j(x_j)), \gamma_i^u(|u_i|)\right\} \\ \Rightarrow &\nabla V_i(x_i) f_i(x, u_i) \le -\alpha_i(V_i(x_i)) \quad \text{a.e.} \end{aligned} \tag{3}$$

where α_i is continuous and positive definite.

In this paper, "a.e." stands for "almost everywhere". The functions γ_{ij}, γ_i^u are known as the ISS gains of the subsystems. With respect to the smooth

Lyapunov characterization introduced by [45], we use only locally Lipschitz ISS-Lyapunov functions because of our constructions in (5), (8), (9), (41) and (134). The following theorem in [25] presents a cyclic-small-gain condition to guarantee the ISS property of the large-scale system (1) with state x and input $u = [u_1^T, \dots, u_N^T]^T$.

Theorem 1. *Consider the large-scale system* (1). *Assume each x_i-subsystem admits an ISS-Lyapunov function V_i satisfying* (2) *and* (3). *Then, the large-scale nonlinear system* (1) *is ISS if for $r = 2, \dots, N$,*

$$\gamma_{i_1 i_2} \circ \gamma_{i_2 i_3} \circ \dots \circ \gamma_{i_r i_1} < \mathrm{Id} \tag{4}$$

where $1 \le i_k \le N$ and $i_k \ne i_{k'}$ if $k \ne k'$ for $1 \le k \le r$.

By considering the subsystems (1) as vertices and the gains as the weights of the directed connections between the subsystems, the interconnection structure of the large-scale nonlinear system can be represented with a system digraph. Condition (4) is called cyclic-small-gain condition and means that the composition of the ISS gains along every simple cycle in the large-scale nonlinear system is less than the identity function Id.

For the ISS gains γ_{ij}'s ($1 \le i \le N$, $j \ne i$) satisfying condition (4), according to [22, Lemma A.1], we can find $\mathcal{K}_\infty$ functions $\hat{\gamma}_{ij}$'s ($1 \le i \le N$, $j \ne i$) which are continuously differentiable on $(0, \infty)$ and slightly larger than the corresponding γ_{ij}'s such that condition (4) still holds by replacing the γ_{ij}'s with the $\hat{\gamma}_{ij}$'s. Motivated by the ISS-Lyapunov function construction in [22], an ISS-Lyapunov function can be constructed for the large-scale system (1) as

$$V(x) = \max_{i=1,\dots,N} \{\sigma_i(V_i(x_i))\} \tag{5}$$

where σ_i's are specific compositions of the $\hat{\gamma}_{(\cdot)}$'s.

The influence of the external input u can be represented as

$$\theta(u) = \max_{i=1,\dots,N} \{\sigma_i \circ \gamma_i^u(|u_i|)\}. \tag{6}$$

Denote $f(x, u) = [f_1^T(x, u_1), \dots, f_N^T(x, u_N)]^T$. With the Lyapunov-based ISS cyclic-small-gain theorem presented in [32], we have

$$V(x) \ge \theta(u) \Rightarrow \nabla V(x) f(x, u) \le -\alpha(V(x)) \quad \text{a.e.} \tag{7}$$

with α being a continuous and positive definite function.

Based on the Lyapunov-based ISS cyclic-small-gain theorem, we have the following lemma.

Lemma 1. *Consider the large-scale system* (1). *Assume that each x_i-subsystem admits an ISS-Lyapunov function V_i satisfying* (2) *and* (3), *and the cyclic-small-gain condition* (4) *is satisfied. Then, V defined in* (5) *is an ISS-Lyapunov* *function of the large-scale system, and at the same time, for any nonempty* $A, B \subseteq \{1, \ldots, N\}$ *satisfying* $A \cup B = \{1, \ldots, N\}$ *and* $A \cap B = \emptyset$,

$$V_A(x_A) = \max_{i \in A}\{\sigma_i(V_i(x_i))\} \tag{8}$$

$$V_B(x_B) = \max_{j \in B}\{\sigma_j(V_j(x_j))\} \tag{9}$$

are ISS-Lyapunov functions of the two systems composed of the x_i*-subsystems with* $i \in A$ *and the* x_j*-subsystems with* $j \in B$*, respectively, where* x_A *and* x_B *represent the states of the two subsystems. Moreover, the interconnection gains can be made less than* Id.

The proof of Lemma 1 is omitted due to space limitation. The interested reader may consult [36, Section 4] for related discussions. Also see Lemma 2 for more details on the interconnection gains.

3 Event-Trigger Design with Partial State Information

In this section, we study an event-trigger design problem for interconnected nonlinear systems. The objective is to develop an ISS gain condition for event-triggered control to avoid infinitely fast sampling and achieve asymptotic convergence.

3.1 Problem Formulation

We consider the case in which a well-designed control system takes the following feedback form:

$$\dot{z}(t) = h(z(t), x(t), w(t)) \tag{10}$$

$$\dot{x}(t) = f(x(t), z(t), w(t)) \tag{11}$$

where $[z^T, x^T]^T$ with $z \in \mathbb{R}^m$ and $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^n$ represents the sampling error of x, $h : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ represent the system dynamics with $h(0, 0, 0) = 0$ and $f(0, 0, 0) = 0$. Here, z is considered to be unavailable for the event-trigger design. For convenience of notations, define $\bar{x} = [z^T, x^T]^T$.

The sampling error $w(t)$ is defined as

$$w(t) = x(t_k) - x(t), \quad t \in [t_k, t_{k+1}),\ k \in \mathbb{S} \tag{12}$$

where $\{t_k\}_{k \in \mathbb{S}}$ is the sequence of the sampling time instants with $\mathbb{S}$ being the set of the indices of all the sampling time instants. Suppose that $\bar{x}(t)$ is right maximally defined for all $t \in [0, T_{\max})$ with $0 < T_{\max} \leq \infty$. With respect to the possible finite-time accumulation of t_k and finite-time escape of $\bar{x}(t)$, we consider three cases:

(a) $\mathbb{S} = \mathbb{Z}_+$ and $\lim_{k\to\infty} t_k < \infty$, which means Zeno behavior.

(b) $\mathbb{S} = \mathbb{Z}_+$ and $\lim_{k\to\infty} t_k = \infty$. In this case, $\bar{x}(t)$ is defined on $[0, \infty)$.

(c) $\mathbb{S}$ is a finite set $\{0, \ldots, k^*\}$ with $k^* \in \mathbb{Z}_+$, i.e., there is a finite number of sampling time instants. In this case, $t_{k^*} < T_{\max}$ and we set $t_{k^*+1} = T_{\max}$ for convenience of discussions.

It should be noted that, in any case, $\bar{x}(t)$ is defined for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$. With an appropriate event trigger design, we can guarantee that $\inf_{k\in\mathbb{S}}\{t_{k+1} - t_k\} > 0$, which means that Case (a) is impossible. Also, as shown in the following discussions, by means of small-gain arguments, we can prove that $T_{\max} = \infty$ for Case (c).

In event-triggered control, the sampling time instants are triggered by comparing $|w(t)|$ with a threshold signal $\mu(t)$ as

$$t_{k+1} = \inf_{t>t_k} \{|w(t)| = \mu(t)\}, \quad k \in \mathbb{S}. \tag{13}$$

Clearly,

$$|w(t)| \leq \mu(t) \tag{14}$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$.

In this paper, the event-trigger design problem is solved by finding an appropriate threshold signal $\mu(t)$ for the event-trigger such that the following two objectives are achieved at the same time:

Objective 1: $z(t)$ and $x(t)$ are defined for all $t \geq 0$ and asymptotically converge to zero.

Objective 2: For any $(z(0), x(0))$ and any $\mu(0) > 0$, the intervals between the sampling time instants are strictly larger than zero.

Here, Objective 2 aims to avoid infinitely fast sampling.

We assume that both the z-subsystem and the x-subsystem are ISS. More precisely, we make the following assumption on the Lyapunov-based ISS properties of the subsystems.

Assumption 1. Both the z-subsystem and the x-subsystem are ISS with ISS-Lyapunov functions $V_z : \mathbb{R}^m \to \mathbb{R}_+$ and $V_x : \mathbb{R}^n \to \mathbb{R}_+$ which are locally Lipschitz on $\mathbb{R}^m\backslash\{0\}$ and $\mathbb{R}^n\backslash\{0\}$, respectively, and satisfy

$$\underline{\alpha}_z(|z|) \leq V_z(z) \leq \overline{\alpha}_z(|z|), \tag{15}$$

$$V_z(z) \geq \max\{\gamma_z^x(V_x(x)), \gamma_z^w(|w|)\}$$
$$\Rightarrow \nabla V_z(z) h(z, x, w) \leq -\alpha_z(V_z(z)), \quad \text{a.e.} \tag{16}$$

$$\underline{\alpha}_x(|x|) \leq V_x(x) \leq \overline{\alpha}_x(|x|), \tag{17}$$

$$V_x(x) \geq \max\{\gamma_x^z(V_z(z)), \gamma_x^w(|w|)\}$$
$$\Rightarrow \nabla V_x(x) f(x, z, w) \leq -\alpha_x(V_x(x)), \quad \text{a.e.} \tag{18}$$

where $\underline{\alpha}_z, \overline{\alpha}_z, \underline{\alpha}_x, \overline{\alpha}_x \in \mathcal{K}_\infty$ and $\gamma_z^x, \gamma_z^w, \gamma_x^z, \gamma_x^w \in \mathcal{K} \cup \{0\}$.

We employ Example 1 to show how an event-triggered control system can be transformed into the form of (10)–(11) satisfying Assumption 1. The system in Example 1 is also used to show the need of a threshold signal which does not decrease exponentially.

Example 1. Consider system

$$\dot{z}(t) = -z^3(t) \tag{19}$$

$$\dot{x}(t) = u(t) + z(t) \tag{20}$$

where $z \in \mathbb{R}$ and $x \in \mathbb{R}$ are the state variables, $u \in \mathbb{R}$ is the control input. We consider the case where only x is available for feedback control design.

We employ the feedback control law

$$u(t) = -x(t_k), \quad t \in [t_k, t_{k+1}), \quad k \in \mathbb{S} \tag{21}$$

where $\{t_k\}_{k\in\mathbb{S}}$ represents the sequence of sampling time instants.

By using (12) and (21), we have

$$\dot{x}(t) = -x(t) - w(t) + z(t). \tag{22}$$

Thus, the controlled system composed of (19) and (22) is in the form of (10)–(11) with $h(z, x, w) = -z^3$ and $f(x, z, w) = -x - w + z$.

To verify the satisfaction of Assumption 1, we define $V_z(z) = |z|$ and $V_x(x) = |x|$. Clearly, V_z and V_x are locally Lipschitz. It can be directly checked that V_z and V_x satisfy (15) and (17), respectively, with $\underline{\alpha}_z, \overline{\alpha}_z, \underline{\alpha}_x, \overline{\alpha}_x = \mathrm{Id}$. Also, direct calculation yields:

$$\nabla V_z(z) h(z, x, w) = -|z|^3 = -V_z^3(z) \quad \text{a.e.} \tag{23}$$

$$\begin{aligned} \nabla V_x(x) f(x, z, w) &\leq -|x| + |w| + |z| \\ &= -V_x(x) + V_z(z) + |w| \\ &\leq -V_x(x) + 2\max\{V_z(z), |w|\} \quad \text{a.e.} \end{aligned} \tag{24}$$

Then, property (24) implies

$$\begin{aligned} &V_x(x) \geq 4\max\{V_z(z), |w|\} \\ &\Rightarrow \nabla V_x(x) f(x, z, w) \leq -0.5 V_x(x) \quad \text{a.e.} \end{aligned} \tag{25}$$

Thus, properties (16) and (18) are satisfied with $\gamma_z^x(s) = 0$, $\gamma_z^w(s) = 0$, $\alpha_z(s) = s^3$, $\gamma_x^z(s) = 4s$, $\gamma_x^w(s) = 4s$ and $\alpha_x(s) = 0.5s$ for $s \in \mathbb{R}_+$.

Under Assumption 1, the interconnected system (10)–(11) is ISS with w as the input if it satisfies the small-gain condition in [24, 22]:

$$\gamma_x^z \circ \gamma_z^x < \mathrm{Id}. \tag{26}$$

As a result, if $w(t)$ asymptotically converges to the origin, then $(z(t), x(t))$ converges to the origin.

In [13, 43], the event-triggered control problem was studied and exponentially converging threshold signals were used in the context of distributed control. Based on this idea, we first try the threshold signal $\mu(t)$ defined by

$$\mu(t) = \mu(0)e^{-ct} \tag{27}$$

for all $t \geq 0$, with initial state $\mu(0) > 0$ and constant $c > 0$. Equivalently, $\mu(t)$ is the solution of the initial value problem

$$\dot{\mu}(t) = -c\mu(t) \tag{28}$$

for all $t \geq 0$.

However, Example 2 shows that an exponentially converging $\mu(t)$ may lead to infinitely fast sampling.

Example 2. Consider the system composed of (19) and (22), which is in the form of (10)–(11) with $h(z, x, w) = -z^3$ and $f(x, z, w) = -x - w + z$. It is shown in Example 1 that the system satisfies Assumption 1. Moreover, the system is a cascade connection of the z-subsystem and the x-subsystem, and thus the small-gain condition (26) is satisfied automatically. We show that for some initial states, there does not exist an exponentially converging threshold signal in the form of (27) to avoid infinitely fast sampling.

Based on the discussions in A, for any specific $z(0), x(0), \mu(0)$ and constant c, there exist constants m_1, m_2, m_3, c^* such that

$$|f(x(t), z(t), w(t))| \geq m_1 e^{-t} + m_2 e^{-c^* t} - m_3 e^{-ct} \tag{29}$$

for all $t \in \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$. Moreover, there exist $z(0), x(0), \mu(0)$ such that

$$m_1 \geq 0,\ m_2 > 0,\ m_2 \geq 2m_3,\ 2c^* \leq c, \tag{30}$$
$$x(t) > z(t) > 0, \tag{31}$$
$$f(x(t), z(t), w(t)) < 0 \tag{32}$$

for all $t \in \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$. Properties (29) and (30) together imply

$$|f(x(t), z(t), w(t))| \geq \frac{m_2}{2} e^{-c^* t} \tag{33}$$

for all $t \in \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$.

Given t_k, we estimate the upper bound of $\delta t_k = t_{k+1} - t_k$. With property (32), we have

$$\begin{aligned} \mu(t_{k+1}) &= \left| \int_{t_k}^{t_{k+1}} f(x(\tau), z(\tau), w(\tau)) d\tau \right| \\ &= \int_{t_k}^{t_{k+1}} |f(x(\tau), z(\tau), w(\tau))| d\tau. \end{aligned} \tag{34}$$

Then, by using (27) and (33), we have

$$\mu(0)e^{-c(t_k+\delta t_k)} \geq \frac{m_2}{2} e^{-c^*(t_k+\delta t_k)}\delta t_k, \tag{35}$$

which implies

$$\delta t_k e^{c^*(t_k+\delta t_k)} \leq \frac{2\mu(0)}{m_2} \tag{36}$$

by using $2c^* \leq c$.

Now, we show $\mathbb{S} = \mathbb{Z}_+$. Suppose $\mathbb{S} = \{0, 1, \dots, k^*\}$ with k^* being a positive integer. In this case, by using the standard small-gain theorem in [24], we can guarantee the boundedness of $z(t)$ and $x(t)$ for all $t \in [0, T_{\max})$. Due to continuation, $z(t)$ and $x(t)$ are defined for all $t \in [0, \infty)$, i.e., $T_{\max} = \infty$. Since the z-subsystem is globally asymptotically stable at the origin, one can find a finite time $t^* \geq t_{k^*}$ such that

$$z(t) \leq \frac{1}{2} x(t_{k^*}) \tag{37}$$

and thus

$$\dot{x}(t) = -x(t_{k^*}) + z(t) \leq -\frac{1}{2} x(t_{k^*}) \tag{38}$$

for all $t \in [t^*, \infty)$. This contradicts with the boundedness of $x(t)$. Thus, $\mathbb{S} = \mathbb{Z}_+$.

Suppose that infinitely fast sampling does not occur. Then, $\lim_{k\to\infty} t_k = \infty$ and $\inf_{k\in\mathbb{Z}_+} \delta t_k > 0$. However, if $\lim_{k\to\infty} t_k = \infty$, then property (36) implies $\lim_{k\to\infty} \delta t_k = 0$. Thus, infinitely fast sampling occurs.

Note that the system dynamics are assumed to be known in Example 2. The problem would be more complicated for nonlinear uncertain systems. In particular, in the setting of event-triggered control, because the sampling error w is updated on discrete time instants (which depend on x and μ), the forward completeness of the system may not be trivially guaranteed.

From the discussions in Example 2, it can be observed that the problem is caused by the nonlinearity z^3 of the z-subsystem. Intuitively, the signal $z(t)$ does not converge to the origin exponentially, and the exponential convergence of $\mu(t)$ is too fast compared with the converging rate of $|f(x(t), z(t), w(t))|$.

To overcome the limitation of the exponentially decreasing threshold signal, we consider threshold signals generated by more general dynamic systems in the form of

$$\dot{\mu}(t) = -\Omega(\mu(t)) \tag{39}$$

with initial condition $\mu(0) > 0$, where $\Omega : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz on compact sets and positive definite. Clearly, (28) is a special case of (39).

Under Assumption 1, we develop a condition on the ISS gains of the subsystems under which event-triggered control can be realized without infinitely fast sampling.

We consider the interconnected system composed of the z-subsystem (10), the x-subsystem (11) and the μ-subsystem (39) subject to (14). Under Assumption 1, if $w(t)$ is well defined for all $t \geq 0$ and the small-gain condition (26) is satisfied, then the interconnected system is asymptotically stable at the origin. Moreover, according to the Lyapunov-based ISS cyclic-small-gain theorem in [32], we can construct a Lyapunov function for the interconnected system:

$$V_0(z, x, \mu) = \max\left\{\hat{\gamma}_x^z(V_z(z)), V_x(x), \hat{\gamma}_x^w(\mu), \hat{\gamma}_x^z \circ \hat{\gamma}_z^w(\mu)\right\}. \tag{40}$$

If $\gamma_{(\cdot)}^{(\cdot)}$ is nonzero, then the corresponding $\hat{\gamma}_{(\cdot)}^{(\cdot)}$ in (40) is chosen such that $\hat{\gamma}_{(\cdot)}^{(\cdot)} \in \mathcal{K}_\infty$ and it is continuously differentiable on $(0, \infty)$ and slightly larger than its corresponding $\gamma_{(\cdot)}^{(\cdot)}$; if $\gamma_{(\cdot)}^{(\cdot)} = 0$, then $\hat{\gamma}_{(\cdot)}^{(\cdot)} = 0$. Moreover, $\hat{\gamma}_x^z$ satisfies $\hat{\gamma}_x^z \circ \gamma_z^x < \mathrm{Id}$.

Define $\breve{\gamma}_x^w(s) = \max\{\hat{\gamma}_x^w(s), \hat{\gamma}_x^z \circ \hat{\gamma}_z^w(s)\}$ for $s \in \mathbb{R}_+$. Clearly, $\breve{\gamma}_x^w$ is a $\mathcal{K}_\infty$ function being locally Lipschitz on $(0, \infty)$. It is a standard result that

$$\begin{aligned} V(z, x, \mu) &= \left(\breve{\gamma}_x^w\right)^{-1}(V_0(z, x, \mu)) \\ &= \max\left\{\left(\breve{\gamma}_x^w\right)^{-1} \circ \hat{\gamma}_x^z(V_z(z)), \left(\breve{\gamma}_x^w\right)^{-1}(V_x(x)), \mu\right\} \\ &:= \max\left\{\sigma_z(V_z(z)), \sigma_x(V_x(x)), \mu\right\} \end{aligned} \tag{41}$$

is also a Lyapunov function of the interconnected system. Note that σ_z and σ_x are locally Lipschitz on $(0, \infty)$ and thus continuously differentiable almost everywhere on $(0, \infty)$.

Moreover, Lemma 2 shows that the interconnection ISS gains are less than Id if $\bar{V}_z(z) = \sigma_z(V_z(z))$ and $\bar{V}_x(x) = \sigma_x(V_x(x))$ are considered as the ISS-Lyapunov functions of the z-subsystem and the x-subsystem, respectively.

Lemma 2. *Under Assumption 1, there exist $\mathcal{K}_\infty$ functions $\bar{\gamma}_z^x, \bar{\gamma}_z^w, \bar{\gamma}_x^z, \bar{\gamma}_x^w$, all less than* Id*, such that*

$$\begin{aligned} &\bar{V}_z(z) \geq \max\{\bar{\gamma}_z^x(\bar{V}_x(x)), \bar{\gamma}_z^w(|w|)\} \\ \Rightarrow &\nabla \bar{V}_z(z) h(z, x, w) \leq -\bar{\alpha}_z(\bar{V}_z(z)) \quad a.e. \end{aligned} \tag{42}$$

$$\begin{aligned} &\bar{V}_x(x) \geq \max\{\bar{\gamma}_x^z(\bar{V}_z(z)), \bar{\gamma}_x^w(|w|)\} \\ \Rightarrow &\nabla \bar{V}_x(x) f(x, z, w) \leq -\bar{\alpha}_x(\bar{V}_x(x)) \quad a.e. \end{aligned} \tag{43}$$

where $\bar{\alpha}_z$ can be any continuous and positive definite function satisfying $\bar{\alpha}_z(s) \leq \partial\sigma_z(\sigma_z^{-1}(s))\alpha_z(\sigma_z^{-1}(s))$ for almost all $s > 0$ and $\bar{\alpha}_x$ can be any continuous and positive definite function satisfying $\bar{\alpha}_x(s) \leq \partial\sigma_x(\sigma_x^{-1}(s))\alpha_x(\sigma_x^{-1}(s))$ for almost all $s > 0$.

Proof. We only prove property (42). Property (43) can be proved similarly.

Since $\sigma_z \in \mathcal{K}_\infty$, $\bar{V}_z$ is positive definite and radially unbounded. We choose $\bar{\gamma}_z^x, \bar{\gamma}_z^w \in \mathcal{K}_\infty$ such that

$$\left(\breve{\gamma}_x^w\right)^{-1} \circ \hat{\gamma}_x^z \circ \gamma_z^x \circ \breve{\gamma}_x^w \leq \bar{\gamma}_z^x < \mathrm{Id}, \tag{44}$$

$$\left(\hat{\gamma}_z^w\right)^{-1} \circ \gamma_z^w \leq \bar{\gamma}_z^w < \mathrm{Id}. \tag{45}$$

The existence of such $\bar{\gamma}_z^x$ and $\bar{\gamma}_z^w$ can be guaranteed as $(\breve{\gamma}_x^w)^{-1} \circ \hat{\gamma}_x^z \circ \gamma_z^x \circ \breve{\gamma}_x^w < (\breve{\gamma}_x^w)^{-1} \circ \breve{\gamma}_x^w = \mathrm{Id}$ and $(\breve{\gamma}_x^w)^{-1} \circ \gamma_z^w < \mathrm{Id}$.

Then, $\bar{V}_z(z) \geq \bar{\gamma}_z^x(\bar{V}_x(x))$ implies $\bar{V}_z(z) \geq (\breve{\gamma}_x^w)^{-1} \circ \hat{\gamma}_x^z \circ \gamma_z^x \circ \breve{\gamma}_x^w(\bar{V}_x(x))$. By using the definitions of $\bar{V}_z$ and $\bar{V}_x$, we have $(\breve{\gamma}_x^w)^{-1} \circ \hat{\gamma}_x^z(V_z(z)) \geq (\breve{\gamma}_x^w)^{-1} \circ \hat{\gamma}_x^z \circ \gamma_z^x \circ \breve{\gamma}_x^w \circ (\breve{\gamma}_x^w)^{-1}(V_x(x))$, which implies $V_z(z) \geq \gamma_z^x(V_x(x))$. Also, $\bar{V}_z(z) \geq \bar{\gamma}_z^w(|w|)$ implies $(\breve{\gamma}_x^w)^{-1} \circ \hat{\gamma}_x^z(V_z(z)) \geq (\hat{\gamma}_z^w)^{-1} \circ \gamma_z^w(|w|)$ and thus $V_z(z) \geq \hat{\gamma}_x^z \circ \breve{\gamma}_x^w \circ (\hat{\gamma}_z^w)^{-1} \circ \gamma_z^w(\mu) \leq \gamma_z^w(|w|)$. It is proved that

$$\begin{aligned} &\bar{V}_z(z) \geq \max\{\bar{\gamma}_z^x(\bar{V}_x(x)), \bar{\gamma}_z^w(|w|)\} \\ \Rightarrow &V_z(z) \geq \max\{\gamma_z^x(V_x(x)), \gamma_z^w(|w|)\}. \end{aligned} \tag{46}$$

Property (42) can then be proved by using (16). This ends the proof of Lemma 2. □

It is shown in the following discussions that, to avoid infinitely fast sampling and at the same time, to guarantee $\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1}) = [0, \infty)$, the decreasing rate of $\mu(t)$ should be chosen in accordance with the decreasing rate of $V(z(t), x(t), \mu(t))$. We first study the decreasing rate of $V(z(t), x(t), \mu(t))$ in Subsection 3.2.

3.2 Decreasing Rate of $V(z(t), x(t), \mu(t))$

According to the definition of V in (41), the decreasing rate of $V(z(t), x(t), \mu(t))$ depends on the decreasing rates of $V_z(z(t))$, $V_x(x(t))$ and $\mu(t)$. Lemma 3 gives a condition on Ω under which the lower bound of the decreasing rate of $V(z(t), x(t), \mu(t))$ can be estimated by Ω.

Lemma 3. *Consider the interconnected system composed of* (10)*,* (11)*,* (12) *and* (39) *subject to* (14)*. Under Assumption 1, if* (26) *is satisfied, and if Ω is Lipschitz on compact sets and positive definite, then there exists a continuous, locally Lipschitz and positive definite α_V such that for any $V(z(0), x(0), \mu(0))$,*

$$V(z(t), x(t), \mu(t)) \leq \eta(t) \tag{47}$$

holds for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$, where $\eta(t)$ is the solution of the initial value problem

$$\dot{\eta}(t) = -\alpha_V(\eta(t)) \tag{48}$$

with initial condition $\eta(0) = V(z(0), x(0), \mu(0))$. Moreover, if there exists a constant $\Delta > 0$ such that

(A) Ω satisfies

$$\Omega(s) \leq \min\left\{\partial\sigma_z(\sigma_z^{-1}(s))\alpha_z(\sigma_z^{-1}(s)), \partial\sigma_x(\sigma_x^{-1}(s))\alpha_x(\sigma_x^{-1}(s))\right\} \tag{49}$$

for almost all $s \in (0, \Delta)$ with σ_z and σ_x defined in (41)*, and*

(B) there exists a $T^O \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$ *such that* $V(z(t), x(t), \mu(t)) \leq \Delta$ *for all* $T^O \leq t < \sup\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$,

then (47) *holds for all* $T^O \leq t < \sup\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$ *with* $\eta(t)$ *being the solution of the initial value problem defined by* (48) *with* $\alpha_V = \Omega$ *for* $T^O \leq t < \sup\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$ *with initial condition* $\eta(T^O) = V(z(T^O), x(T^O), \mu(T^O))$.

Proof. The result of (47) can be proved by using a combination of the Lyapunov-based cyclic-small-gain theorem and the comparison principle, and is omitted here. See, e.g., [26, Lemma 3.4], for the comparison principle.

Suppose that Conditions (A) and (B) are satisfied. Then, (42) and (43) hold with

$$\bar{\alpha}_z = \bar{\alpha}_x = \Omega \tag{50}$$

for all $[z^T, x^T, \mu]^T$ satisfying $V(z, x, \mu) < \Delta$.

For convenience of discussions, define $v_1(t) = \bar{V}_z(z(t))$, $v_2(t) = \bar{V}_x(x(t))$, $v_3(t) = \mu(t)$, and $v(t) = V(z(t), x(t), \mu(t))$. Then $v(t) = \max\{v_1(t), v_2(t), v_3(t)\}$.

Now, we prove

$$D^+ v(t) \leq -\Omega(v(t)) \tag{51}$$

for all $T^O \leq t < \sup\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$, where D^+ represents the upper right-hand derivative and is defined by

$$D^+ v(t) = \limsup_{h\to 0^+} \frac{v(t+h) - v(t)}{h}. \tag{52}$$

Consider a specific time instant t satisfying $T^O \leq t < \sup\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$. Since $\bar{\gamma}_z^x < \mathrm{Id}$ and $\bar{\gamma}_z^w < \mathrm{Id}$, when $v_1(t) = v(t) > 0$, $\bar{V}_z(z(t)) = V(z(t), x(t), \mu(t)) > 0$, there exists a neighborhood Θ of $[z^T(t), x^T(t), \mu(t)]^T$ such that $\bar{V}_z(p_z) \geq \max\{\bar{\gamma}_z^x(\bar{V}_x(p_x)), \bar{\gamma}_z^w(p_\mu)\}$ and $\nabla V_z(p_z)$ exists for all $[p_z^T, p_x^T, p_\mu] \in \Theta$. Then, due to the continuity of $\bar{V}_z(z(t)), \bar{V}_x(x(t))$ with respect to t, there exists a $t' > t$ such that $\bar{V}_z(z(\tau)) \geq \max\{\bar{\gamma}_z^x(\bar{V}_x(x(\tau))), \bar{\gamma}_z^w(|\mu(\tau)|)\} \geq \max\{\bar{\gamma}_z^x(\bar{V}_x(x(\tau))), \bar{\gamma}_z^w(|w(\tau)|)\}$ for all $\tau \in (t, t')$, which implies

$$\nabla \bar{V}_z(z(\tau)) h(z(\tau), x(\tau), w(\tau)) \leq -\Omega(\bar{V}_z(z(\tau))) \tag{53}$$

for all $\tau \in (t, t')$, according to (42). Then, we have

$$D^+ v_1(t) \leq -\Omega(v_1(t)). \tag{54}$$

Following the same reasoning, if $v_2(t) = v(t)$, then

$$D^+ v_2(t) \leq -\Omega(v_2(t)). \tag{55}$$

Note that $D^+ v_3(t) = -\Omega(v_3(t))$ holds automatically for $v_3(t) = \mu(t)$.

For $T^O \leq t < \sup \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$, define $I(t) = \{i \in \{1, 2, 3\} : v_i(t) = v(t)\}$. Then, by using [11, Lemma 2.9], we have

$$\begin{aligned} D^+ v(t) &= \max\{D^+ v_i(t) : i \in I(t)\} \\ &\leq \max\{-\Omega(v_i(t)) : i \in I(t)\} \\ &= -\Omega(v(t)). \end{aligned} \tag{56}$$

Property (51) is proved.

Then, by directly applying the comparison principle, the proof of Lemma 3 follows readily. □

Remark 1. Condition (A) in Lemma 3 is concerned with Ω. Given specific $\partial\sigma_z(s)\alpha_z(\sigma_z^{-1}(s))$ and $\partial\sigma_x(s)\alpha_x(\sigma_x^{-1}(s))$, one can always find a Ω satisfying the condition. In the following subsection, we will show how Condition (B) can be satisfied by appropriately choosing Ω for the event-triggered control system.

3.3 Event-Trigger Design

The main result of this section is given in Theorem 2.

Theorem 2. *Consider the interconnected system composed of* (10)*,* (11)*,* (14) *and* (39) *with Assumption 1 and* (26) *satisfied. Then, for any initial state, Objectives 1 and 2 given in Subsection 3.1 are achievable if*

- Ω *is chosen to be Lipschitz on compact sets and positive definite,*
- *there exists a constant* $\Delta > 0$ *such that* $\Omega(s)/s$ *is nondecreasing for* $s \in (0, \Delta]$ *and* Ω *satisfies* (49) *for almost all* $s \in (0, \Delta)$*, and*
- $(\sigma_z \circ \underline{\alpha}_z)^{-1}$ *and* $(\sigma_x \circ \underline{\alpha}_x)^{-1}$ *are Lipschitz on compact sets.*

Proof. Due to the positive definiteness of Ω, the $\mu(t)$ generated by (39) satisfies

$$0 \leq \mu(t) \leq \mu(0) \tag{57}$$

for all $t \geq 0$. Moreover, since Ω is chosen to be Lipschitz on compact sets, there exists a constant $\bar{c} > 0$ such that

$$\Omega(s) \leq \bar{c}s \tag{58}$$

for $0 \leq s \leq \mu(0)$, and thus

$$\dot{\mu}(t) = -\Omega(\mu(t)) \geq -\bar{c}\mu(t) \tag{59}$$

along the trajectory of μ with initial state $\mu(0)$. A direct application of the comparison principle implies

$$\mu(\tau) \geq \mu(t)e^{-\bar{c}(\tau - t)} \tag{60}$$

for all $\tau \geq t \geq 0$.

Also, by using the event-trigger (13), we have

$$\begin{aligned}\mu(t_{k+1}) &= |x(t_{k+1}) - x(t_k)| \\ &= \left| \int_{t_k}^{t_{k+1}} f(x(\tau), z(\tau), w(\tau)) d\tau \right| \\ &\le \int_{t_k}^{t_{k+1}} |f(x(\tau), z(\tau), w(\tau))| \, d\tau. \end{aligned} \tag{61}$$

If the conditions of Theorem 2 are satisfied, then with Lemma 3, the function V defined in (41) has property (47) for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$ with $\eta(t)$ being generated by (48). Due to the positive definiteness of α_V, for any initial condition $V(z(0), x(0), \mu(0))$,

$$V(z(t), x(t), \mu(t)) \le V(z(0), x(0), \mu(0)) \tag{62}$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$.

If $V(z(0), x(0), \mu(0)) \le \Delta$, then define $T^* = 0$; otherwise, define T^* as the first time instant such that $\eta(T^*) = \Delta$, where $\eta(t)$ is the solution of the initial value problem (48) with $\eta(0) = V(z(0), x(0), \mu(0))$. Recall that, according to Lemma 3, property (47) holds for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$.

Let us estimate the lower bound of $\delta t_k = t_{k+1} - t_k$ by considering the cases of $t_k \le T^*$ and $t_k > T^*$ separately.

Case 1: $t_k \le T^*$. In this case, we prove that given specific $z(0), x(0), \mu(0)$ and specific T^*, there exists a $\delta_0 > 0$ such that $\delta t_k \ge \delta_0$. In this way, we can also guarantee that $z(t)$, $x(t)$ and $w(t)$ are defined for all $t \in [0, T^*]$ and thus $T^* \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$.

Property (62) means that there exists a finite $\Delta_s > 0$ depending on the initial state such that

$$\left|[z^T(t), x^T(t), \mu(t)]^T\right| \le \Delta_s \tag{63}$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$. Thus, there exists a Δ_f such that

$$|f(z(t), x(t), w(t))| \le \Delta_f \tag{64}$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$. Then, by also using properties (60) and (61), we have

$$\begin{aligned}\mu(0)e^{-\bar{c}(t_k+\delta t_k)} &\le \int_{t_k}^{t_{k+1}} |f(x(\tau), z(\tau), w(\tau))| \, d\tau \\ &\le (t_{k+1} - t_k)\Delta_f = \delta t_k \Delta_f, \end{aligned} \tag{65}$$

i.e.,

$$\delta t_k e^{\bar{c}(t_k+\delta t_k)} \ge \frac{\mu(0)}{\Delta_f}. \tag{66}$$

If $t_k \leq T^*$, it is concluded that

$$\delta t_k e^{\bar{c}(T^* + \delta t_k)} \geq \frac{\mu(0)}{\Delta_f}. \tag{67}$$

Then, δ_0 can be chosen to satisfy $\delta_0 e^{\bar{c}(T^* + \delta_0)} = \mu(0)/\Delta_f$.

Case 2: $t_k > T^*$**.** In this case, we prove that given specific $z(T^*)$, $x(T^*)$ and $\mu(T^*)$, there exists a $\delta_1 > 0$ such that $\delta t_k \geq \delta_1$.

In this case, by using (47) and the definition of T^*, we have

$$V(z(t), x(t), \mu(t)) \leq \Delta \tag{68}$$

for all $T^* \leq t < \sup \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$.

Consider an $\eta_1(t)$ defined by

$$\dot{\eta}_1(t) = -\Omega(\eta_1(t)) \tag{69}$$

for all $t > T^*$ with $\eta_1(T^*) = V(z(T^*), x(T^*), \mu(T^*))$. Then, by using Lemma 3, we have $V(z(t), x(t), \mu(t)) \leq \eta_1(t)$ for all $T^* < t < \sup \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$. Also, by using the definition of V in (41), we have $V(z(t), x(t), \mu(t)) \geq \mu(t)$ for all $t \in \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$. Thus,

$$\mu(t) \leq V(z(t), x(t), \mu(t)) \leq \eta_1(t) \tag{70}$$

for all $T^* < t < \sup \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$.

With a similar reasoning as for (60), it can be proved that the $\eta_1(t)$ defined by (69) is strictly positive for all $T^* < t < \sup \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$.

Define

$$k_\mu = \frac{\eta_1(T^*)}{\mu(T^*)}. \tag{71}$$

Then, according to (70), $k_\mu \geq 1$. We prove that

$$\eta_1(t) \leq k_\mu \mu(t) \tag{72}$$

for all $T^* < t < \sup \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$.

Since $\Omega(s)/s$ is non-decreasing for all $s \in (0, \Delta]$, we have

$$\frac{\Omega(\eta_1)}{\eta_1} \geq \frac{\Omega(\eta_1/k_\mu)}{\eta_1/k_\mu}, \tag{73}$$

which implies $\Omega(\eta_1)/k_\mu \geq \Omega(\eta_1/k_\mu)$ for $\eta_1 \in (0, \Delta]$. Then, by using (69), we have

$$\frac{1}{k_\mu}\dot{\eta}_1(t) = -\frac{1}{k_\mu}\Omega(\eta_1(t)) \leq -\Omega\left(\frac{1}{k_\mu}\eta_1(t)\right) \tag{74}$$

for all $T^* < t < \sup \bigcup_{k \in \mathbb{S}} [t_k, t_{k+1})$. Property (72) can then be proved by using the comparison principle for $\eta_1(t)/k_\mu$ and $\mu(t)$.

If $(\sigma_z \circ \underline{\alpha}_z)^{-1}$ and $(\sigma_x \circ \underline{\alpha}_x)^{-1}$ are Lipschitz on compact sets, then one can find constants $k_z, k_x > 0$ such that

$$\eta_1(t) \geq V(z(t), x(t), \mu(t)) \geq \max\{k_z(|z(t)|), k_x(|x(t)|)\}. \tag{75}$$

for all $T^* < t < \sup \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$. Then, property (72) implies

$$\mu(t) \geq \frac{1}{k_\mu} \max\{k_z(|z(t)|), k_x(|x(t)|)\} \tag{76}$$

for all $T^* < t < \sup \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$.

By using the locally Lipschitz property of f, there exists a constant $k_f > 0$ such that

$$|f(z, x, w)| \leq k_f \max\{|z|, |x|, \mu\} \tag{77}$$

for all (z, x, μ) satisfying $V(z, x, \mu) \leq V(z(T^*), x(T^*), \mu(T^*))$.

Then, properties (61), (76) and (77) together imply

$$\begin{aligned}\mu(t_{k+1}) &\leq (t_{k+1} - t_k)k_f \max_{t_k\leq\tau\leq t_{k+1}} \{|z(\tau)|, |x(\tau)|, \mu(\tau)\}\\ &\leq (t_{k+1} - t_k)k_f \max_{t_k\leq\tau\leq t_{k+1}} \{k_\mu\mu(\tau)/k_z, k_\mu\mu(\tau)/k_x, \mu(\tau)\}\\ &\leq \delta t_k k_f \max\{k_\mu/k_z, k_\mu/k_x, 1\}\,\mu(t_k).\end{aligned} \tag{78}$$

Also note that (60) means

$$\mu(t_{k+1}) \geq e^{-\bar{c}\delta t_k}\mu(t_k). \tag{79}$$

Thus, we have

$$e^{-\bar{c}\delta t_k} \leq \delta t_k k_f \max\{k_\mu/k_z, k_\mu/k_x, 1\}, \tag{80}$$

i.e.,

$$\delta t_k e^{\bar{c}\delta t_k} \geq k_f \max\{k_\mu/k_z, k_\mu/k_x, 1\}. \tag{81}$$

Then, δ_1 can be chosen such that $\delta_1 e^{\bar{c}\delta_1} \geq k_f \max\{k_\mu/k_z, k_\mu/k_x, 1\}$.

We now consider the above-mentioned three cases: (a) $\mathbb{S} = \mathbb{Z}_+$ and $\lim_{k\to\infty} t_k < \infty$; (b) $\mathbb{S} = \mathbb{Z}_+$ and $\lim_{k\to\infty} t_k = \infty$; (c) $\mathbb{S} = \{0, \ldots, k^*\}$ with $k^* \in \mathbb{Z}_+$.

(1) Suppose that Case (a) happens. Then, we have $\inf_{k\in\mathbb{S}}\{t_{k+1} - t_k\} = 0$, which contradicts with (67) and (81). Thus, Case (a) is impossible.

(2) In Case (b), we have $\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1}) = [0, \infty)$, which means that (47) holds for all $t \in [0, \infty)$.

(3) In Case (c), since $t_{k^*+1} = T_{\max}$, we have $\bigcup_{k\in\mathbb{S}}[t_k, t_{k+1}) = [0, T_{\max})$, and thus (47) holds for all $t \in [0, T_{\max})$. By the continuation of solutions, this implies that $\bar{x}(t)$ is defined for all $t \in [0, \infty)$, i.e., $T_{\max} = \infty$.

This ends the proof of Theorem 2. □

Remark 2. For $s, t \geq 0$, define $\beta(s,t) = \eta(t)$ with $\eta(t)$ being the solution of the initial value problem (48) with initial condition $\eta(0) = s$. It can be directly checked that $\beta \in \mathcal{KL}$. Recall that the proof of Theorem 2 shows that for any $z(0) \in \mathbb{R}^m$, any $x(0) \in \mathbb{R}^n$ and any $\mu(0) > 0$, (47) holds for all $t \geq 0$, which implies

$$V(z(t), x(t), \mu(t)) \leq \beta(V(z(0), x(0), \mu(0)), t) \tag{82}$$

for all $t \geq 0$. Then, because of the positive definiteness and radial unboundedness of V defined in (41), there exists a $\beta' \in \mathcal{KL}$ such that for any $z(0) \in \mathbb{R}^m$, any $x(0) \in \mathbb{R}^n$ and any $\mu(0) > 0$,

$$|[z^T(t), x^T(t), \mu(t)]^T| \leq \beta'(|[z^T(0), x^T(0), \mu(0)]^T|, t) \tag{83}$$

for all $t \geq 0$. Thus, $\mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}_+ \backslash \{0\}$ is a region of attraction of the closed-loop event-triggered system with $[z^T, x^T, \mu]^T$ as the state.

Example 3. The infinitely fast sampling problem arising in Example 2 can be readily solved by Theorem 2. By using the V_z and V_x defined in Example 1, we choose $\hat{\gamma}_x^z(s) = 5s$, $\hat{\gamma}_z^w(s) = 0$ and $\hat{\gamma}_x^w(s) = 5s$ for $s \in \mathbb{R}_+$. According to (40), we define

$$V_0(z, x, \mu) = \max\{5V_z(z), V_x(x), 5\mu\}. \tag{84}$$

By choosing $\breve{\gamma}_x^w(s) = 5s$ for $s \in \mathbb{R}_+$, we have

$$V(z, x, \mu) = \max\{V_z(z), V_x(x)/5, \mu\}. \tag{85}$$

Thus, $\sigma_z(s) = s$ and $\sigma_x(s) = s/5$ for $s \in \mathbb{R}_+$.

It can be verified that $(\sigma_z \circ \underline{\alpha}_z)^{-1}(s) = s$ and $(\sigma_x \circ \underline{\alpha}_x)^{-1}(s) = 2s$ for $s \in \mathbb{R}_+$ are Lipschitz on compact sets.

We choose

$$\begin{aligned}\Omega(s) &= \min\left\{\partial\sigma_z(s)\alpha_z(\sigma_z^{-1}(s)), \partial\sigma_x(s)\alpha_x(\sigma_x^{-1}(s))\right\} \\ &= \min\{s^3, s/2\}\end{aligned} \tag{86}$$

for $s \in \mathbb{R}_+$. Then, Ω is positive definite and locally Lipschitz, and satisfies (49). Also, $\Omega(s)/s$ is non-decreasing for $s \in (0, \infty)$. A simulation example is given in Section 5.

Remark 3. A special case is that both the z-subsystem and the x-subsystem are linear and their ISS-Lyapunov functions are in the quadratic form. In this case, Assumption 1 can be modified with $\underline{\alpha}_z(s) = \underline{a}_z s^2$, $\overline{\alpha}_z(s) = \overline{a}_z s^2$, $\underline{\alpha}_x(s) = \underline{a}_x s^2$, $\overline{\alpha}_x(s) = \overline{a}_x s^2$, $\gamma_z^x(s) = b_z^x s$, $\gamma_z^w(s) = b_z^w s^2$, $\gamma_x^z(s) = b_x^z s$, $\gamma_x^w(s) = b_x^w s^2$, $\alpha_z(s) = a_z s$ and $\alpha_x(s) = a_x s$ for $s \in \mathbb{R}_+$ with $\underline{a}_z, \overline{a}_z, \underline{a}_x, \overline{a}_x, b_z^x, b_z^w, b_x^z, b_x^w, a_z, a_z$ being positive constants.

For the linear case, the small-gain condition (26) is equivalent to $b_x^z b_z^x < 1$. Assume that the small-gain condition is satisfied. Then, there exists an $\epsilon > 0$ such that $(b_x^z + \epsilon)b_z^x < 1$. We choose $\hat{\gamma}_x^z(s) = (b_x^z + \epsilon)s := \hat{b}_x^z s$, $\hat{\gamma}_x^w(s) = (b_x^w + \epsilon)s^2 := \hat{b}_x^w s^2$, $\hat{\gamma}_z^w(s) = (b_z^w + \epsilon)s := \hat{b}_z^w s^2$ for $s \in \mathbb{R}_+$. Then, $\breve{\gamma}_x^w$ can be written in the form $\breve{\gamma}_x^w(s) = \breve{b}_x^w s^2$ with $\breve{b}_x^w$ being a positive constant. It can be calculated that $\sigma_z(s) = \sqrt{\hat{b}_x^z s/\breve{b}_x^w}$ and $\sigma_x(s) = \sqrt{s/\breve{b}_x^w}$. Then, the right-hand side of (49) equals $\min\left\{a_z/\hat{b}_x^z, a_x\right\}\breve{b}_x^w s/2$. Thus, one can find a positive constant c such that $\Omega(s) = cs$ satisfies (49).

Remark 4. The event-trigger design in this section considers the systems which are input-to-state stabilizable with the sampling error w as the input. This is closely related to the topic of control under sensor noise if we consdier the sampling error as a special sensor noise. One of the previous results can be found in [21], which considers nonlinear systems composed of two subsystems, one is ISS and the other one is input-to-state stabilizable with respect to the sensor noise. In [21], the ISS of the closed-loop system is guaranteed by using the ISS small-gain theorem in [24, 22]. In Section 4, we show that the proposed condition for event-triggered control can also be satisfied by a class of nonlinear uncertain systems through a new nonlinear output-feedback control design.

3.4 An Extension

In the discussions above, we consider $\mu(t)$ to be generated by system (39) with the system dynamics depending solely on $\mu(t)$. A more general case is that $\mu(t)$ is generated by a system

$$\dot{\mu}(t) = -\bar{\Omega}(\mu(t), x(t)) \tag{87}$$

with $\bar{\Omega} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ being an appropriately chosen function and $\mu(0) > 0$. In this case, the structure of the interconnected system composed of (10), (11) and (87) subject to (14) is shown in Figure 1.

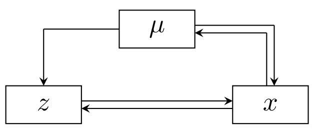

Figure 1: The structure of the interconnected system composed of (10), (11) and (87).

Under Assumption 1, the basic idea is to choose $\bar{\Omega}$ such that

- for all $\mu \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$,

$$\bar{\Omega}(\mu, x) \le \Omega(\mu) \tag{88}$$

  where Ω is chosen to satisfy the conditions given in Theorem 2;

- system (87) is ISS with μ as the state and x as the input, and moreover,

$$\mu \geq \chi_\mu^x(|x|) \Rightarrow \bar{\Omega}(\mu, x) \geq \alpha_\mu(\mu) \tag{89}$$

where α_μ is a continuous, positive definite function and χ_μ^x is a $\mathcal{K}$ function and satisfies

$$\gamma_x^w \circ \chi_\mu^x \circ \underline{\alpha}_x^{-1} < \mathrm{Id}. \tag{90}$$

Then, the system composed of subsystems (10), (11) and (87) subject to (14) can be considered as an interconnection of ISS subsystems, and conditions (26) and (90) form the cyclic-small-gain condition given in [25] for the interconnected system. If (26) and (90) are satisfied, then the interconnected system is ISS.

With $\mu(t)$ generated by (87), one can still guarantee the boundedness of $\mu(t)$ for all $t \in \bigcup_{k \in \mathbb{S}}[t_k, t_{k+1})$. And with a similar reasoning as for (59), one can find a $\bar{c} > 0$ such that

$$\dot{\mu}(t) \geq -\bar{c}\mu(t) \tag{91}$$

for all $t \in \bigcup_{k \in \mathbb{S}}[t_k, t_{k+1})$. Then, the validity of (87) can be proved in the same way as in the proof of Theorem 2. Note that in this case, Lemma 3 should also be generalized with (48) replaced by

$$\dot{\eta}(t) = -\bar{\Omega}(\eta(t), x(t)). \tag{92}$$

One realization of the $\bar{\Omega}$ satisfying (88) and (89) is

$$\bar{\Omega}(\mu, x) = \Omega(\mu) - \chi_1\left(\max\left\{\chi_2 \circ \chi_\mu^x(|x|) - \chi_2(\mu), 0\right\}\right) \tag{93}$$

where Ω is chosen to satisfy the conditions given in Theorem 2, χ_1 and χ_2 can be any $\mathcal{K}$ function. Clearly, with such design, condition (89) is satisfied with $\alpha_\mu = \Omega$.

4 Event-Triggered Output-Feedback Control

The discussions in Section 3 are based on Assumption 1. In this section, we show how the condition is satisfied for an event-triggered output-feedback control system.

We consider the popular class of nonlinear uncertain systems in the output-feedback form [27]:

$$\dot{x}_i = x_{i+1} + \Delta_i(x_1), \quad i = 1, \ldots, n-1 \tag{94}$$
$$\dot{x}_n = u + \Delta_n(x_1) \tag{95}$$
$$y = x_1 \tag{96}$$

where $x := [x_1, \ldots, x_n]^T$ with $x_i \in \mathbb{R}$ is the state, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control input, and $\Delta_1, \ldots, \Delta_n : \mathbb{R} \to \mathbb{R}$ are uncertain functions. It is assumed that only y is available for feedback.

The following assumption is made on $\Delta_1, \ldots, \Delta_n$.

Assumption 2. For each $i = 1, \dots, n$, there exists a known $\psi_{\Delta_i} \in \mathcal{K}_\infty$ such that

$$|\Delta_i(r)| \leq \psi_{\Delta_i}(|r|) \tag{97}$$

for all $r \in \mathbb{R}$.

Based on the achievements in Section 3, the objective of this section is to design an event-triggered output-feedback controller for system (94)–(96) so that the closed-loop system is in the form of (10)–(11) and moreover the conditions for event-triggered control are satisfied.

4.1 Observer-Based Output-Feedback Controller

For convenience of notations, we define

$$y^m = y + w \tag{98}$$

where y^m represents the sampled value of y and w is the sampling error. In the setting of event-triggered control, only the sampled data of y can be used for feedback control.

Specifically, owing to the output-feedback structure, we design a nonlinear observer for system (94)–(96) as

$$\dot{\xi}_1 = \xi_2 + L_2\xi_1 + \rho_1(\xi_1 - y^m) \tag{99}$$

$$\dot{\xi}_i = \xi_{i+1} + L_{i+1}\xi_1 - L_i(\xi_2 + L_2\xi_1), \quad 2 \leq i \leq n-1 \tag{100}$$

$$\dot{\xi}_n = u - L_n(\xi_2 + L_2\xi_1) \tag{101}$$

where $\rho_1 : \mathbb{R} \to \mathbb{R}$ is an odd and strictly decreasing function, and $L_2, \dots, L_n$ are positive constants. In the observer, ξ_1 is an estimate of y, and ξ_i is an estimate of $x_i - L_i y$ for $2 \leq i \leq n$. For convenience of discussions, we define observation errors:

$$\zeta_1 = y - \xi_1, \tag{102}$$

$$\zeta_i = x_i - L_i y - \xi_i, \quad i = 2, \dots, n. \tag{103}$$

Based on the estimation by the observer, we design a nonlinear control law in the following form:

$$e_1 = y \tag{104}$$

$$e_2 = \xi_2 - \kappa_1(e_1 - \zeta_1) \tag{105}$$

$$e_i = \xi_i - \kappa_{i-1}(e_{i-1}), \quad i = 3, \dots, n \tag{106}$$

$$u = \kappa_n(e_n) \tag{107}$$

where $\kappa_1, \dots, \kappa_n$ are continuously differentiable, odd, strictly decreasing and radially unbounded functions. Note that $e_1 - \zeta_1 = y - \zeta_1 = \xi_1$. Thus, the control law uses only the state of the observer $(\xi_1, \dots, \xi_n)$, and is realizable.

The desired system structure is shown in Figure 2, which is in accordance with the general structure given by [3, Fig. 2].

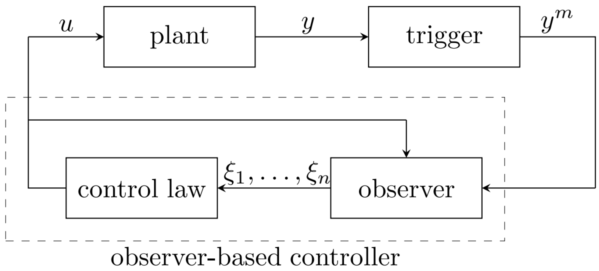

Figure 2: The structure of the event-triggered output-feedback control system.

Remark 5. Equation (99) of the observer is constructed to estimate y by using the available y^m. The function ρ_1 in (99) is used to assign an appropriate, probably *nonlinear*, gain to the observation error system, more precisely, equation (111), to satisfy the cyclic-small-gain condition. Equations (100)–(101) are in the same spirit of the reduced-order observer in [23]. Slightly differently, we use ξ_1 instead of the unavailable y in (100)–(101).

4.2 ISS Property of the Subsystems

In this subsection, we show that there exist $\rho_1, L_2, \ldots, L_n, \kappa_1, \ldots, \kappa_n$ to transform the closed-loop system into a network of ISS subsystems. In particular, we prove that the subsystems with $\zeta_1, \bar{\zeta}_2, e_1, \ldots, e_n$ as the states are ISS, where $\bar{\zeta}_2 = [\zeta_2, \ldots, \zeta_n]^T$. For the subsystems, we define the following ISS-Lyapunov function candidates:

$$V_{\zeta_1}(\zeta_1) = |\zeta_1|, \tag{108}$$

$$V_{\bar{\zeta}_2}(\bar{\zeta}_2) = \left(\bar{\zeta}_2^T P \bar{\zeta}_2\right)^{\frac{1}{2}}, \tag{109}$$

$$V_{e_i}(e_i) = |e_i|, \quad i = 1, \ldots, n. \tag{110}$$

The matrix P for $V_{\bar{\zeta}_2}$ is positive definite and is determined later.

Based on the result in this subsection, in the following subsection, an ISS cyclic-small-gain method is employed to ultimately transform the closed-loop system into the form of (10)–(11) with the conditions for event-triggered control satisfied.

4.2.1 The ζ_1-subsystem

By taking the derivative of ζ_1, we have

$$\dot{\zeta}_1 = \rho_1(\zeta_1 + w) + \phi_1(\zeta_1, \zeta_2, e_1) \tag{111}$$

where

$$\phi_1(\zeta_1, \zeta_2, e_1) = L_2\zeta_1 + \zeta_2 + \Delta_1(e_1). \tag{112}$$

It can be directly verified that there exists a $\psi_{\phi_1} \in \mathcal{K}_\infty$ being Lipschitz on compact sets such that $|\phi_1(\zeta_1, \zeta_2, e_1)| \leq \psi_{\phi_1}(|[\zeta_1, \zeta_2, e_1]^T|)$.

With Lemma 4 in the Appendix, one can find a continuously differentiable ρ_1 such that for any constant $0 < c < 1$, $\ell_{\zeta_1} > 0$ and any $\chi_{\zeta_1}^{\zeta_2}, \chi_{\zeta_1}^{e_1} \in \mathcal{K}$ which are Lipschitz on compact sets, the ζ_1-subsystem is ISS with $V_{\zeta_1}(\zeta_1)$ as an ISS-Lyapunov function, which satisfies

$$
\begin{aligned}
& V_{\zeta_1}(\zeta_1) \geq \max\left\{\chi_{\zeta_1}^{\zeta_2}(V_{\bar{\zeta}_2}(\bar{\zeta}_2)), \chi_{\zeta_1}^{e_1}(V_{e_1}(e_1)), \chi_{\zeta_1}^{w}(|w|)\right\} \\
\Rightarrow & \nabla V_{\zeta_1}(\zeta_1)\left(\rho_1(\zeta_1 + w) + \phi_1(\zeta_1, \zeta_2, e_1)\right) \leq -\ell V_{\zeta_1}(\zeta_1)
\end{aligned} \tag{113}
$$

for almost all ζ_1, where

$$
\chi_{\zeta_1}^{w}(s) = \frac{s}{c} \tag{114}
$$

for $s \in \mathbb{R}_+$.

4.2.2 The $\bar{\zeta}_2$-subsystem

By taking the derivative of $\bar{\zeta}_2$, we can write the $\bar{\zeta}_2$-subsystem as

$$
\dot{\bar{\zeta}}_2 = A\bar{\zeta}_2 + \bar{\phi}_2(\zeta_1, e_1) \tag{115}
$$

where

$$
A = \begin{bmatrix} -L_2 & & & \\ \vdots & & I_{n-1} & \\ -L_{n-1} & & & \\ -L_n & 0 & \cdots & 0 \end{bmatrix} \tag{116}
$$

$$
\bar{\phi}_2(\zeta_1, e_1) = \begin{bmatrix} \phi_{i2}(\zeta_1, e_1) \\ \vdots \\ \phi_n(\zeta_1, e_1) \end{bmatrix} \tag{117}
$$

with

$$
\begin{aligned}
\phi_i(\zeta_1, e_1) &= (L_{i+1} - L_i L_2)\zeta_1 - L_i\Delta_1(e_1) + \Delta_i(e_1), \\
& \qquad i = 2, \ldots, n-1 && (118) \\
\phi_n(\zeta_1, e_1) &= -L_n L_2 \zeta_1 - L_n \Delta_1(e_1) + \Delta_n(e_1). && (119)
\end{aligned}
$$

By choosing $L_2, \ldots, L_n$ such that A is Hurwitz, there exists a positive definite matrix $P = P^T \in \mathbb{R}^{(n-1)\times(n-1)}$ satisfying $PA + A^T P = -2I_{n-1}$. Define $V_{\bar{\zeta}_2}^0(\bar{\zeta}_2) = \bar{\zeta}_2^T P \bar{\zeta}_2$. Then, there exist $\underline{\alpha}_{\bar{\zeta}_2}^0, \overline{\alpha}_{\bar{\zeta}_2}^0 \in \mathcal{K}_\infty$ such that $\underline{\alpha}_{\bar{\zeta}_2}^0(|\bar{\zeta}_2|) \leq V_{\bar{\zeta}_2}^0(\bar{\zeta}_2) \leq \overline{\alpha}_{\bar{\zeta}_2}^0(|\bar{\zeta}_2|)$. With direct calculation, we have

$$
\begin{aligned}
\nabla V_{\bar{\zeta}_2}^0(\bar{\zeta}_2)\dot{\bar{\zeta}}_2 &= -2\bar{\zeta}_2^T\bar{\zeta}_2 + 2\bar{\zeta}_2^T P_i \bar{\phi}_2(\zeta_1, e_1) \\
&\leq -\bar{\zeta}_2^T\bar{\zeta}_2 + |P|^2|\bar{\phi}_2(\zeta_1, e_1)|^2 \\
&\leq -\frac{1}{\lambda_{\max}(P)}V_{\bar{\zeta}_2}^0(\bar{\zeta}_2) + |P|^2\left(\psi_{\bar{\phi}_2}^{\zeta_1}(|\zeta_1|) + \psi_{\bar{\phi}_2}^{e_1}(|e_1|)\right),
\end{aligned} \tag{120}
$$

where $\breve{\psi}^{\zeta_1}_{\bar{\phi}_2}$ and $\breve{\psi}^{e_1}_{\bar{\phi}_2}$ are $\mathcal{K}_\infty$ functions such that

$$|\bar{\phi}_2(\zeta_1, e_1)|^2 \le \breve{\psi}^{\zeta_1}_{\bar{\phi}_2}(|\zeta_1|) + \breve{\psi}^{e_1}_{\bar{\phi}_2}(|e_1|) \tag{121}$$

for all $\zeta_1, e_1 \in \mathbb{R}$. Moreover, $\breve{\psi}^{\zeta_1}_{\bar{\phi}_2}$ and $\breve{\psi}^{e_1}_{\bar{\phi}_2}$ can be written as $\breve{\psi}^{\zeta_1}_{\bar{\phi}_2} = \psi^{\zeta_1}_{\bar{\phi}_2}(s^2)$ and $\breve{\psi}^{e_1}_{\bar{\phi}_2} = \psi^{e_1}_{\bar{\phi}_2}(s^2)$ for $s \in \mathbb{R}_+$, with $\psi^{\zeta_1}_{\bar{\phi}_2}$ and $\psi^{e_1}_{\bar{\phi}_2}$ being Lipschitz on compact sets.

This means that the $\bar{\zeta}_2$-subsystem is ISS with $V^0_{\bar{\zeta}_2}$ as an ISS-Lyapunov function. The ISS gains can be chosen as follows. Define $\breve{\chi}^{\zeta_1}_{\bar{\zeta}_2} = 3\lambda_{\max}(P)|P^2|\psi^{\zeta_1}_{\bar{\phi}_2}$ and $\breve{\chi}^{e_1}_{\bar{\zeta}_2} = 3\lambda_{\max}(P)|P^2|\psi^{e_1}_{\bar{\phi}_2}$. Then,

$$\begin{aligned}
&V^0_{\bar{\zeta}_2}(\bar{\zeta}_2) \ge \max\left\{\breve{\chi}^{\zeta_1}_{\bar{\zeta}_2}(V_{\zeta_1}(\zeta_1)), \breve{\chi}^{e_1}_{\bar{\zeta}_2}(V_{e_1}(e_1))\right\} \\
\Rightarrow &\nabla V^0_{\bar{\zeta}_2}(\bar{\zeta}_2)\left(A\bar{\zeta}_2 + \bar{\phi}_2(\zeta_1, e_1)\right) \le -\ell_{\bar{\zeta}_2} V^0_{\bar{\zeta}_2}(\bar{\zeta}_2)
\end{aligned} \tag{122}$$

where $\ell_{\bar{\zeta}_2} = 1/3\lambda_{\max}(P_i)$.

Hence, there exist $\chi^{\zeta_1}_{\bar{\zeta}_2}, \chi^{e_1}_{\bar{\zeta}_2} \in \mathcal{K}$ being Lipschitz on compact sets and a continuous, positive definite $\alpha_{\bar{\zeta}_2}$ such that

$$\begin{aligned}
&V_{\bar{\zeta}_2}(\bar{\zeta}_2) \ge \max\left\{\chi^{\zeta_1}_{\bar{\zeta}_2}(V_{\zeta_1}(\zeta_1)), \chi^{e_1}_{\bar{\zeta}_2}(V_{e_1}(e_1))\right\} \\
\Rightarrow &\nabla V_{\bar{\zeta}_2}(\bar{\zeta}_2)\left(A\bar{\zeta}_2 + \bar{\phi}_2(\zeta_1, e_1)\right) \le -\alpha_{\bar{\zeta}_2}(V_{\bar{\zeta}_2}(\bar{\zeta}_2)) \quad \text{a.e.}
\end{aligned} \tag{123}$$

4.2.3 The e_i-subsystems ($i = 1, \dots, n$)

It can also be proved that the $(e_1, \dots, e_n)$-subsystem can be derived into the form

$$\dot{e}_1 = \kappa_1(e_1 - \zeta_1) + \varphi_1(e_1, e_2, \zeta_2) \tag{124}$$
$$\dot{e}_2 = \kappa_2(e_2) + \varphi_2(e_1, e_2, e_3, \zeta_1, w) \tag{125}$$
$$\dot{e}_i = \kappa_i(e_i) + \varphi_i(e_1, e_2, \dots, e_{i+1}, \zeta_1), \quad i = 3, \dots, n. \tag{126}$$

For convenience of notations, we denote $\dot{e}_1 = h_1(e_1, e_2, \zeta_1, \zeta_2)$, $\dot{e}_2 = h_2(e_1, e_2, e_3, \zeta_1, w)$ and $\dot{e}_i = h_i(e_1, \dots, e_{i+1}, \zeta_1)$ for $i = 3, \dots, n$.

Moreover, there exist $\psi_{\varphi_1}, \dots, \psi_{\phi_n} \in \mathcal{K}_\infty$ being Lipschitz on compact sets such that

$$|\varphi_1(e_1, e_2, \zeta_2)| \le \psi_{\varphi_1}(|[e_1, e_2, \zeta_2]^T|) \tag{127}$$
$$|\varphi_2(e_1, e_2, e_3, \zeta_1, w)| \le \psi_{\varphi_2}(|[e_1, e_2, e_3, \zeta_1, w]^T|) \tag{128}$$
$$|\varphi_i(e_1, e_2, \dots, e_i, \zeta_1)| \le \psi_{\varphi_i}(|[e_1, e_2, \dots, e_i, \zeta_1]^T|), \quad i = 3, \dots, n. \tag{129}$$

With Lemma 4 in B, we can find continuously differentiable κ_i for $i = 1, \dots, n$ such that each e_i-subsystem is ISS with $V_{e_i}(e_i) = |e_i|$ as an ISS-

Lyapunov function. Specifically, we have

$$\begin{aligned}&V_{e_1}(e_1) \geq \max\left\{\chi_{e_1}^{e_2}(V_{e_2}(e_2)), \chi_{e_1}^{\zeta_1}(V_{\zeta_1}(\zeta_1)), \chi_{e_1}^{\bar{\zeta}_2}(V_{\bar{\zeta}_2}(\bar{\zeta}_2))\right\}\\ \Rightarrow&\nabla V_{e_1}(e_1)h_1(e_1, e_2, \zeta_1, \zeta_2) \leq -\ell_{e_1}V_{e_1}(e_1), \quad \text{a.e.}\end{aligned} \tag{130}$$

$$\begin{aligned}&V_{e_2}(e_2) \geq \max\left\{\chi_{e_2}^{e_1}(V_{e_1}(e_1)), \chi_{e_2}^{e_3}(V_{e_3}(e_3)), \chi_{e_2}^{\zeta_1}(V_{\zeta_1}(\zeta_1)), \chi_{e_2}^{w}(|w|)\right\}\\ \Rightarrow&\nabla V_{e_2}(e_2)h_2(e_1, e_2, e_3, \zeta_1, w) \leq -\ell_{e_2}V_{e_2}(e_2), \quad \text{a.e.}\end{aligned} \tag{131}$$

and for $i = 3, \ldots, n$,

$$\begin{aligned}&V_{e_i}(e_i) \geq \max_{j=1,\ldots,i-1,i+1}\left\{\chi_{e_i}^{e_j}(V_{e_j}(e_j)), \chi_{e_i}^{\zeta_1}(V_{\zeta_1}(\zeta_1))\right\}\\ \Rightarrow&\nabla V_{e_i}(e_i)h_i(e_1, \ldots, e_{i+1}, \zeta_1) \leq -\ell_{e_i}V_{e_i}(e_i), \quad \text{a.e.}\end{aligned} \tag{132}$$

where the $\ell_{(\cdot)}$'s can be any specified positive constants, $\chi_{e_n}^{e_n+1} = 0$,

$$\chi_{e_1}^{\zeta_1}(s) = \frac{s}{c} \tag{133}$$

for $s \in \mathbb{R}_+$, where c can be chosen to be any constant satisfying $0 < c < 1$, and the other $\chi_{(\cdot)}^{(\cdot)}$'s can be any $\mathcal{K}_\infty$ functions which are Lipschitz on compact sets.

4.3 Event-Triggered Control

With the nonlinear observer-based controller, the closed-loop system has been transformed into a network of ISS subsystems with $\zeta_1, \bar{\zeta}_2, e_1, \ldots, e_n$ as the states.

In this subsection, we rewrite the $(\zeta_1, \bar{\zeta}_2, e_1, \ldots, e_n)$-system into the form of (10)–(11) and design an event-triggered controller without infinitely fast sampling. Since only the output y, i.e., e_1, is available to the event-trigger, we consider the $(\zeta_1, \bar{\zeta}_2, e_2, \ldots, e_n)$-subsystem as subsystem (10) and consider the e_1-subsystem as subsystem (11). For convenience of notations, we define $z = [\zeta_1, \bar{\zeta}_2^T, e_2, \ldots, e_n]^T$ and denote $\dot{z} = g(z, e_1, w)$.

The main result on event-triggered output-feedback control is given by Theorem 3.

Theorem 3. *Under Assumption 2, the closed-loop system composed of* (94)–(96), (99)–(101) *and* (104)–(107) *can be transformed into an interconnection of two ISS subsystems with the measurement error $w(t)$ caused by data-sampling as the external input. Moreover, asymptotic stabilization can be achieved through event-triggered control without infinitely fast sampling by using event-trigger* (13) *with the threshold signal $\mu(t)$ generated by* (39) *or more generally* (87).

Proof. We choose the $\chi_{(\cdot)}^{(\cdot)}$'s such that

- the $(\zeta_1, \bar{\zeta}_2, e_1, \ldots, e_n)$-system satisfies the cyclic-small-gain condition;
- each $\chi_{(\cdot)}^{(\cdot)}$ and the corresponding $\left(\chi_{(\cdot)}^{(\cdot)}\right)^{-1}$ are Lipschitz on compact sets.

Then, we can find $\hat{\chi}^{(\cdot)}_{(\cdot)} \in \mathcal{K}_\infty$ being continuously differentiable on $(0,\infty)$ such that $\hat{\chi}^{(\cdot)}_{(\cdot)}$ and $\left(\hat{\chi}^{(\cdot)}_{(\cdot)}\right)^{-1}$ are Lipschitz on compact sets and the cyclic-small-gain condition is still satisfied if the $\chi^{(\cdot)}_{(\cdot)}$'s are replaced by their corresponding $\hat{\chi}^{(\cdot)}_{(\cdot)}$'s.

Then, with the Lyapunov-based ISS cyclic-small-gain theorem,

$$V(z, e_1) = \max\{V_z(z), V_{e_1}(e_1)\} \tag{134}$$

is an ISS-Lyapunov function of the (z, e_1)-system, where

$$V_z(z) = \max_{i=2,\ldots,n} \left\{\sigma_{\zeta_1}(V_{\zeta_1}(\zeta_1)), \sigma_{\bar{\zeta}_2}(V_{\bar{\zeta}_2}(\bar{\zeta}_2)), \sigma_{e_i}(V_{e_i}(e_i))\right\} \tag{135}$$

with the $\sigma_{(\cdot)}$'s being appropriate compositions of the $\hat{\chi}^{(\cdot)}_{(\cdot)}$'s.

By using Lemma 1, $V_z(z)$ is an ISS-Lyapunov function of the z-subsystem satisfying

$$\begin{aligned} &V_z(z) \geq \max\{\chi_z^{e_1}(V_{e_1}(e_1)), \chi_z^w(|w|)\} \\ \Rightarrow &\nabla V_z(z) g(z, e_1, w) \leq -\alpha_z(V_z(z)) \quad \text{a.e.} \end{aligned} \tag{136}$$

where $\chi_z^{e_1}, \chi_z^w \in \mathcal{K}_\infty$ and α_z is a continuous, positive definite function. Moreover, $\chi_z^{e_1} < \mathrm{Id}$.

Also, by using (135) and (130), we have

$$\begin{aligned} &V_{e_1}(e_1) \geq \chi_{e_1}^z(V_z(z)) \\ \Rightarrow &\nabla V_{e_1}(e_1) h_1(e_1, e_2, \zeta_1, \zeta_2) \leq -\ell_{e_1} V_{e_1}(e_1) \quad \text{a.e.} \end{aligned} \tag{137}$$

where $\chi_{e_1}^z \in \mathcal{K}$ is less than Id.

Clearly, the interconnection of the z-subsystem and the e_1-subsystem also satisfies the small-gain condition, i.e.,

$$\chi_{e_1}^z \circ \chi_z^{e_1} < \mathrm{Id}. \tag{138}$$

The closed-loop system has been transformed into an interconnection of two ISS subsystems (with z and e_1 as the states) satisfying the small-gain condition. Note that all the gain functions and their inverse functions are chosen to be Lipschitz on compact sets. Thus, their compositions are also Lipschitz on compact sets. Then, it can be proved that the third condition in Theorem 2 is also satisfied. This ends the proof of Theorem 3. □

5 A Simulation Example

We use a simulation example to verify the theoretical results. Consider the system given in Example 1. The design of an event-trigger (13) with the threshold signal $\mu(t)$ generated by (39) is given in Example 3. We choose initial conditions: $z(0) = -3$, $x(0) = 1$ and $\mu(0) = 10$.

Figure 3 shows the convergence of $x(t)$ and the convergence of $w(t)$ bounded by the threshold signal $\mu(t)$. The control signal $u(t)$ and the inter-sampling times $\delta t_k = t_k - t_{k-1}$ during the event-triggered control process is shown in Figure 4. According to the simulation, the minimal inter-sampling time for $0 \leq t \leq 300$ is 0.998.

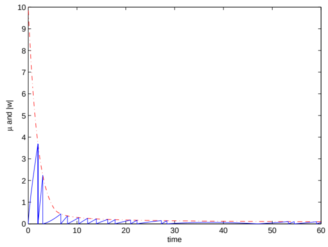

Figure 3: The trajectories of μ and w with $\Omega(\mu) = \min\left\{\mu^3, \mu/2\right\}$.

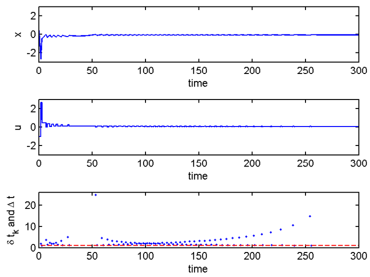

Figure 4: The state x, the control signal u and the inter-sampling times $\delta t_k = t_k - t_{k-1}$ with $\Omega(\mu) = \min\left\{\mu^3, \mu/2\right\}$. The minimal inter-sampling time during the period of simulation is 0.998.

For comparison, we also consider event-triggers with exponentially decreasing threshold signals. Figure 5 shows the inter-sampling times during the control process with $\Omega(\mu) = \mu/2$ and $\Omega(\mu) = \mu/20$, respectively. The step-length of the numerical simulation is chosen to be 0.001 for an acceptable simulation accuracy. For the simulation of the event-triggered control system, the step-length actually guarantees strictly positive minimum inter-sampling times and infinitely fast sampling never happens. However, from Figure 5, we can still observe that the inter-sampling times converge to a very small neighborhood of the zero in finite time. In the recent paper [16], for linear systems, the authors studied the combination of event-triggered control and periodic sampled-data control, which is of practical interest and deserves more effort for nonlinear systems.

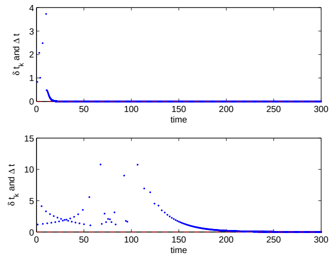

Figure 5: The inter-sampling times $\delta t_k = t_k - t_{k-1}$ with $\Omega(\mu) = \mu/2$ and $\Omega(\mu) = \mu/20$, respectively.

6 Conclusions

This paper has developed a new small-gain approach to event-triggered control of nonlinear uncertain systems with partial state and output feedback. In particular, the event-trigger design problem for the systems that are transformable into an interconnection of two ISS subsystems is solved for the first time. It is shown that infinitely fast sampling can be avoided by considering the threshold signal to be generated by an asymptotically stable system. Based on this result, a more general class of event-triggers with the threshold signals depending on the real-time system state has also been proposed for the first time in the literature. Moreover, the event-triggered output-feedback control problem for nonlinear systems in the output-feedback form has been solved through a novel nonlinear observer-based design. The results of this paper pave a foundation to further investigate the following problems:

- Event-triggered control of nonlinear systems with quantized measurements. In networked control systems, data-sampling and quantization usually coexist. Recently, we have developed small-gain methods for quantized control of nonlinear systems in [37]. In the quantized control results, we use ISS gains to represent the influence of quantization error, while in this paper, we employ an ISS gain to represent the influence of data-sampling. This creates an opportunity to develop a unified framework for event-triggered and quantized control of nonlinear systems.

- Distributed event-triggered control. Small-gain techniques also bridge event-triggered control and our recent distributed control results. In [33], it is shown that a distributed control problem for nonlinear uncertain systems can ultimately be transformed into a stability problem of a network of ISS subsystems. By integrating the idea in this paper, distributed control could be realized through event-triggered information exchange. Note that such idea has been implemented for linear systems in [50, 8, 43, 13].

A More Remarks about Example 2

Define $\upsilon = x - z$ and $v = z^3$. Then,

$$\begin{aligned}\dot{\upsilon}(t) &= \dot{x}(t) - \dot{z}(t) \\ &= -\upsilon(t) - w(t) + z^3(t) \\ &\geq -\upsilon(t) - \mu(t) + v(t),\end{aligned} \tag{139}$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$ and

$$\dot{v}(t) = 3z^2(t)\dot{z}(t) = -3z^5(t) = -3v^{\frac{5}{3}}(t) \tag{140}$$

for all $t \geq 0$.

We consider the case of $v(0) > 0$. In this case, $v(t)$ is strictly decreasing and $v(t) \leq v(0)$ for all $t \geq 0$. Then, one can find a $c^* > 0$ such that

$$v(t) \geq v(0)e^{-c^* t} := \breve{v}(t) \tag{141}$$

for all $t \geq 0$. Moreover, if $v(0) < 1$, then c^* can be chosen to be strictly less than one.

Define $\upsilon^*(t)$ as the solution of the initial value problem

$$\dot{\upsilon}^*(t) = -\upsilon^*(t) - \mu(t) + \breve{v}(t) \tag{142}$$

with initial condition $\upsilon^*(0) = \upsilon(0)$. Then, a direct application of the comparison principle yields:

$$\upsilon(t) \geq \upsilon^*(t) \tag{143}$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k, t_{k+1})$.

With $\mu(t)$ defined in (27) and $\breve{v}(t)$ defined above, if $c \neq 1$, then

$$\begin{aligned}\upsilon^*(t) &= \upsilon(0)e^{-t} + \int_0^t e^{-(t-\tau)}\left(-\mu(\tau) + \breve{v}(t)\right) d\tau \\ &= \upsilon(0)e^{-t} + e^{-t}\int_0^t e^{\tau}\left(-\mu(0)e^{-c\tau} + \breve{v}(0)e^{-c^*\tau}\right) d\tau \\ &= \left(\upsilon(0) + \frac{\mu(0)}{1-c} - \frac{v(0)}{1-c^*}\right) e^{-t} + \frac{v(0)}{1-c^*}e^{-c^* t} \\ &\quad - \frac{\mu(0)}{1-c}e^{-ct}.\end{aligned} \tag{144}$$

Thus,

$$
\begin{aligned}
|f(x(t),z(t),w(t))| &= |-x(t)-w(t)+z(t)| \\
&= |v(t)+w(t)| \\
&\geq v(t)-\mu(t) \\
&\geq \left(v(0)+\frac{\mu(0)}{1-c}-\frac{v(0)}{1-c^*}\right)e^{-t} \\
&\quad + \frac{v(0)}{1-c^*}e^{-c^*t}-\frac{\mu(0)}{1-c}e^{-ct}-\mu(0)e^{-ct} \\
&:= m_1e^{-t}+m_2e^{-c^*t}-m_3e^{-ct} \qquad (145)
\end{aligned}
$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k,t_{k+1})$. Moreover, there exist $z(0)>0, x(0)>0, \mu(0)>0$ such that

$$
m_1 \geq 0,\ m_2>0,\ m_2\geq 2m_3,\ 2c^*\leq c. \qquad (146)
$$

In this case, it is directly checked that

$$
|f(x(t),z(t),w(t))| \geq v(t)-\mu(t) \geq \frac{m_2}{2}e^{-c^*t} > 0 \qquad (147)
$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k,t_{k+1})$. Following a similar reasoning, property (147) can still be proved in the case of $c=1$.

Recall $v(t)=x(t)-z(t)$. With $z(0)>0$ and $\mu(0)>0$, we have $z(t)>0$, $\mu(t)>0$ and thus

$$
x(t) > z(t) > 0 \qquad (148)
$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k,t_{k+1})$.

With $v(t)-\mu(t)>0$ given by (147), we also have

$$
\begin{aligned}
f(x(t),z(t),w(t)) &= -x(t)-w(t)+z(t) \\
&= -v(t)-w(t) \\
&\leq -v(t)+\mu(t) < 0 \qquad (149)
\end{aligned}
$$

for all $t \in \bigcup_{k\in\mathbb{S}}[t_k,t_{k+1})$.

B A Technical Lemma for Gain Assignment

The gain assignment technique plays an important role in designing the output-feedback control system to satisfy the ISS gain conditions for event-triggered control. We give a technical lemma for gain assignment to make the paper self-contained.

Consider a first-order nonlinear system:

$$
\dot{\eta} = \phi(\eta,\omega_1,\dots,\omega_m)+\bar{\kappa} \qquad (150)
$$
$$
\eta^m = \eta+\omega_{m+1} \qquad (151)
$$

where $\eta \in \mathbb{R}$ is the state, $\bar{\kappa} \in \mathbb{R}$ is the control input, $\omega_1, \ldots, \omega_{m+1} \in \mathbb{R}$ represent external inputs, $\eta^m \in \mathbb{R}$ is the measurement of η, the nonlinear function $\phi(\eta, \omega_1, \ldots, \omega_m)$ is locally Lipschitz and satisfies

$$|\phi(\eta, \omega_1, \ldots, \omega_m)| \leq \psi_\phi(|[\eta, \omega_1, \ldots, \omega_m]^T|) \tag{152}$$

with $\psi_\phi \in \mathcal{K}_\infty$. It can be proved that condition (152) implies

$$|\phi(\eta, \omega_1, \ldots, \omega_m)| \leq \psi_\phi^\eta(|\eta|) + \sum_{k=1}^{m} \psi_\phi^{\omega_k}(|\omega_k|) \tag{153}$$

with $\psi_\phi^\eta, \psi_\phi^{\omega_1}, \ldots, \psi_\phi^{\omega_m} \in \mathcal{K}_\infty$.

Lemma 4. *Consider system* (151)*. For any specified* $0 < c < 1$*,* $\ell > 0$ *and* $\gamma_\eta^{\omega_1}, \ldots, \gamma_\eta^{\omega_m} \in \mathcal{K}_\infty$*, we can find a* $\kappa : \mathbb{R} \to \mathbb{R}$ *which is odd and continuously differentiable on* $(-\infty, 0) \cup (0, \infty)$ *and satisfies*

$$\kappa((1-c)s) \geq \psi_\phi^\eta(s) + \sum_{k=1}^{m} \psi_\phi^{\omega_k} \circ \left(\gamma_\eta^{\omega_k}\right)^{-1}(s) + \frac{\ell}{2}s \tag{154}$$

for all $s \geq 0$*, such that the closed-loop system with* $\bar{\kappa} = \kappa(\eta^m)$ *is ISS with* $V_\eta(\eta) = |\eta|$ *as an ISS-Lyapunov function, which satisfies*

$$\begin{aligned} &V_\eta(\eta) \geq \max_{k=1,\ldots,m+1} \left\{\gamma_\eta^{\omega_k}(|\omega_k|)\right\} \\ \Rightarrow\ &\nabla V_\eta(\eta)\left(\phi(\eta, \omega_1, \ldots, \omega_m) + \kappa(\eta^m)\right) \leq -\ell V_\eta(\eta), \quad a.e. \end{aligned} \tag{155}$$

where $\gamma_\eta^{\omega_{m+1}}(s) = s/c$ *for* $s \in \mathbb{R}_+$*. Moreover, if* $\left(\gamma_\eta^{\omega_1}\right)^{-1}, \ldots, \left(\gamma_\eta^{\omega_m}\right)^{-1}$ *are Lipschitz on compact sets, then* κ *can be chosen to be continuously differentiable on* $(-\infty, \infty)$*.*

By considering $V_\eta(\eta) = |\eta|$ as an ISS-Lyapunov function, Lemma 4 can be proved based on the proofs of the gain assignment lemmas in [20, 35]. The proof is not provided here due to space limitation.

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